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A THEORY OF CALCULATION OF FLUTTER VIBRATIONS IN SUBSONIC FLOWS

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A THEORY OF CALCULATION OF FLUTTER VIBRATIONS IN SUBSONIC FLOWS[1]

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ABSTRACT. Calculation of a critical flutter velocity for a membrane in a nonconservative system. It is shown that the damping forces have a destabilizing effect and that a discontinuity appears in the stability criterion which depends on the damping coefficient and is characteristic of nonconservative systems. By means of this theory, the flutter of weather vanes or sails, as well as the related traveling transverse waves, can be explained.

I. Introduction. /22*

The calculation of the flutter oscillations of a thin profile or a plate without flow may be accomplished using the singularity method by means of a fictional vortex layer that changes with time. The calculation is relatively complex and vague; it eventually leads to the problem of the solution of an integral equation. Kuessner [1], Schwarz [2] and Soehngen [3] performed calculations of flutter oscillations in 1936-1940. The calculation would be much simpler if there were a pressure law that would describe the reaction of the flowing medium on a plate element as an explicit function of the deformation of the elements. Such a pressure law would have the advantage that the problem of calculating the flutter oscillations would be separated from concepts of aerodynamics and become a pure problem of technical oscillation theory. Now such a pressure law is available for supersonic flows, /23
provided by Ashley and Zartarian [4] according to the "Piston Theory." One is therefore led to wonder whether or not such a pressure law could be obtained for incompressible flows as well.

[1]The present paper was delivered in abbreviated form by the author at the GAMM Congress in Zuerich (1967), under the title "A Contribution to the Calculation of Flutter Oscillations."
*Numbers in the margin indicate pagination in the foreign text.

II. Derivation of the Pressure Law.

In the following discussion, it will be assumed that we are dealing with incompressible flow which is essentially friction-free. The flow around an oscillating plate will be compared with the flow inside an oscillating tube through which a medium is flowing. Such tubes have recently been studied in connection with their stability [5], and the results of the calculations are also in agreement with the phenomona that occur in technology. In order to understand the forces that are created at the tube wall by the flow velocity v of the medium in the tube and those causing the movement of the tube, see Figure 1. A medium element with relative velocity $v_r = v$ is moving through the oscillating tube element with a deflection $z(x,t)$. In the kinematic sense, the tube element constitutes a vehicular element, toward which a medium element moves with relative velocity v_r. A familiar principle of kinematics [6], however, says in connection with such a relative movement toward a vehicle that the absolute acceleration is composed additively of three components, so that

$$b = b_f + b_r + b_c,$$

where b_f is the vehicular acceleration, b_r is the relative acceleration, and b_c is the Coriolis acceleration. Since in the present case the motion of the tube elements occurs primarily in the z-direction, $b_f = z_{tt}$. The relative acceleration at constant r consists of the relative centripetal acceleration $b_r = v^2 z_{xx}$. The Coriolis acceleration is calculated as follows /24

$$b_c = 2\ (\omega \times v_r),$$

so that with an angular velocity of the vehicle $\omega = z_{xt}$ we will have $b_c = 2\ v\ z_{xt}$. If we let q equal the force exerted by the tube wall on the flowing medium, based on the unit length of the tube axis, the motion equation of the medium element in the direction of the instantaneous perpendicular to the trajectory plane will be

$$q\ dx = \mu_F\ (v^2 z_{xx} + 2\ v\ z_{xt} + z_{tt})\ d\ x,$$

from which we obtain as the reaction of the flowing medium against the tube wall, the line load q as follows:

$$q = \mu_F (v^2 z_{xx} + 2 v z_{xt} + z_{tt}). \tag{II.1}$$

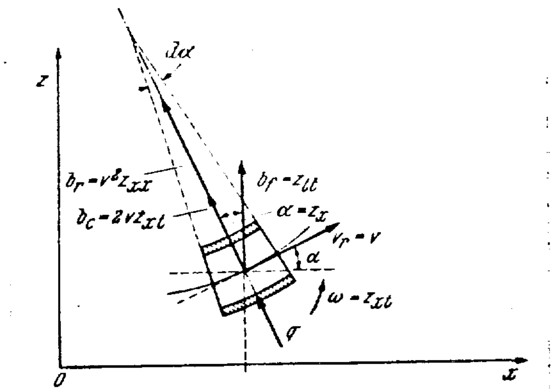

Figure 1. Kinetic Values For a Tube Element with Flow Through It.

In this pressure law, μ_F through the internal cross-section of the tube according to the filamentary flow theory applied here is a known mass of the flowing medium directly involved in the transverse oscillations of the tube, based on the unit length of the tube axis. As far as the tube is concerned, there is no objection to using (II.1), at least when a very thin tube is employed, for which the ratio r/L composed of the tube radius r of the internal cross section and the tube length 1 is much less than 1. A theoretical expansion of (II.1) occurs which is valid with respect to technical applications as well, if the right-hand side of (II.1) is increased by a damping factor, so that we have

$$q = \mu_F (v^2 z_{xx} + 2 v z_{xt} + z_{tt}) + \delta_0 z_t. \tag{II.2}$$

Such an assumption of damping with damping factor δ_0 is necessary in the sense of modern stability theory according to Lyaponov [7], and to a certain degree is even necessary, as we shall see later on in the discussion of the results. The pressure law (II.2) can be applied to a plate around which a medium is flowing if q and μ_F are based on a unit area of the plate and weight per unit area μ_F is assumed known. The pressure law given by the "Piston Theory" for supersonic flows, in comparison with (II.2), is

$$q = \delta_1 v z_x + \delta_0 z_t, \tag{II.3}$$

where $\delta_1 = \delta_0 = \kappa p_\infty / c_\infty$ represents a coefficient determined by the medium; here κ is the polytropic exponent, p_∞ and c_∞ are the pressure and speed of sound in an undisturbed medium. It is advantageous for further calculations, however, to ensure a discussion of the results that is as complete as possible, so that $\delta_1 \neq \delta_0$. In (II.3) the deflection of the plate element is contained in only one term with the derivative z_x; the second term in this instance also represents damping.

III. Calculation of the Critical Velocity for a Membrane with Rectangular Boundaries. /25

We shall now use pressure laws (II.2) and (II.3), for comparison of the results they yeild, in the case of a membrane (Figure 2) with rectangular boundaries subjected to flow in the x-direction at velocity v. The membrane is assumed to be stressed in the x- and y-directions by forces S_1 and S_2 based on the unit length, and these forces are further assumed to be positive, pressure forces. Since the edges of the membrane are assumed fixed, the desired solution z (x, y, t) for the movements of the membrane must satisfy the boundary conditions

$$\begin{aligned} z\,(x, 0, t) &= 0 \qquad & z\,(0, y, t) &= 0 \\ z\,(x, d, t) &= 0 \qquad & z\,(c, y, t) &= 0. \end{aligned} \tag{III.1}$$

In addition, in the sense of the stability study at time t = 0, the initial disturbances

$$z\,(x, y, 0) = \phi\,(x, y) \qquad z_t\,(x, y, 0) = \Phi\,(x,y) \tag{III.2}$$

may be given for the initial deflection and the initial velocity, where the functions ϕ (x, y) and Φ (x, y) are random within rather broad limits. If the pressure law (II.3) is initially applied to the membrane, we will obtain the partial differential equation

$$S_1\, Z_{xx} + S_2\, Z_{yy} + \mu p\, Z_{tt} + \delta_1\, v\, z_x + \delta_0\, z_t = o\,, \tag{III.3}$$

as a motion equation, in which μ_p is the weight per unit area of the membrane. It is not necessary to assume an additional external damping in this equation which is proportional to the deflection velocity z_t; instead, this can be thought of as included in the damping factor δ_0 since $\delta_1 \neq \delta_0$ and the two numbers may therefore be thought of as independent of each other. With the expression

$$z = u(x, t) \sin \frac{\pi m y}{d} \quad (m = 1, 2, 3, \ldots) \tag{III.4}$$

(III.3) is changed to the partial differential equation /26

$$S_1 u_{xx} + \mu\rho u_{tt} + \delta_1 v u_x + \delta_0 u_t - S_2 \frac{\pi^2 m^2}{d^2} u = 0.$$

This equation can then be solved by means of the separation expression

$$u = e^{ixx} e^{i\beta t} \tag{III.5}$$

where, with maintenance of the boundary conditions in x according to (III.1) and summation over all existing particular integrals the following solution can be obtained:

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\frac{\delta_1 v x}{2 S_1}} \sin \frac{\pi n x}{c} \sin \frac{\pi m y}{d} \left(c_{1mn} e^{i\beta_{1mn} t} + c_{2mn} e^{i\beta_{2mn} t}\right) \tag{III.6}$$

Here, c_{1mn} and c_{2mn} are freely selectable constants, and the characteristic exponents of the time are therefore

$$i\beta_{12mn} = -\frac{\delta_0}{2\mu\rho} \pm \sqrt{\frac{\delta_0^2}{4\mu\rho^2} + \frac{\delta_1^2 v^2}{4\mu\rho S_1} + \frac{S_2}{\mu\rho}\left(\frac{\pi m}{d}\right)^2 + \frac{S_1}{\mu\rho}\left(\frac{\pi n}{c}\right)^2}. \tag{III.7}$$

These exponents describe the behavior of equation (III.6) with time. In order to develop the solution z (x, y, t) to satisfy random initial disturbances (III.2) according to (III.6), all particular integrals are required, and hence the exponents (III.7) must be discussed for all combinations of whole--number values of m and n. It is advantageous in conjunction with further calculation to consider separately the two different cases $\delta_0 = 0$ and $\delta_0 > 0$ with arbitrary $\delta_1 \geq 0$. If we initially assume that $\delta_0 = 0$, i.e., if we calculate without damping, it becomes evident that instability is avoided either because no oscillations developed that become apparent with time or (what amounts to the same thing) exponents $i\beta_{12mn}$ exhibit no positive real parts, if

$$S_1 < 0 \text{ and } S_2 < 0 \tag{III.8}$$

for all values of $\delta_1 \geqq 0$. However, if we assume that $\delta_0 < 0$ and thereby assume damping, the requirement for avoidance of instability imposes the condition (III.8), although stability is now assured and a behavior of solution (III.6) exists which decreases with time. Criterion (III.8) therefore requires validity for all combinations of numbers $\delta_0 \geqq 0$ and $\delta_1 \geqq 0$ which are involved. The stability criterion $S_1 < 0$ and $S_2 < 0$ which applies here is, however, actually trivial. It does indicate, however, that the membrane must be stressed in the x- and y-direction only by means of tension in order to avoid instability. From the physical standpoint, it is remarkable and really not very convincing that criterion (III.8) does not contain the flow velocities v, and that there is consequently no critical velocity v_k to cause flutter oscillations of the membrane. Hence, the true principal and different /27
movement conditions are stable or unstable at the membrane plate regardless of the magnitude of the flow velocity v. This result can also be found in Bolotin [8]. When bending resistance is included in the calculation, however, the velocity v of the medium can influence the stability of the plate [4], [9].

The application of pressure law (II.2) to the membrane leads to a solution for the z (x,y,t) which is different from (III.6), and the stability criterion is also structured completely differently from the criterion (III.8). The motion equation for the membrane which is to be solved is then the partial differential equation

$$S_1^* z_{xx} + S_2 z_{yy} + 2 v \mu_F z_{xt} + \mu z_{tt} + \delta_0 z_t = 0 \qquad \text{(III.9)}$$

with

$$S_1^* = S_1 + v^2 \mu_F \qquad \text{(III.10)}$$

and $\mu = \mu_F + \mu_P$ is the sum of the weight per unit area μ_F of the flow medium and the weight per unit area μ_P of the membrane. With the assumption (III.4) which satisfies the boundary conditions for z (x,y,t) in y according to (III.1), we obtain from (III.9) the partial differential equation

$$S_1^x u_{xx} + 2 v \mu_F u_{xt} + \mu u_{tt} + \delta_0 u_t - S_2 \left(\frac{\pi m}{d}\right)^2 u = 0,$$

which can be solved by means of the separation expression (III.5) or directly by means of the expression

$$u(x, t) = F(ax + bt) \sin \frac{\pi n x}{c} \qquad (n = 1, 2, 3, \ldots).$$

Here, F (ax + bt) constitutes a transverse wave moving through the membrane in the x-direction at a velocity

$$w = -\frac{b}{a} = \frac{S_1^*}{v \mu_F}. \tag{III.11}$$

For the function F, we can find the conventional second-order differential equation

$$\frac{d^2 F}{du^2} + \frac{dF}{du} \frac{\delta_0}{\mu b} \frac{S_1^*}{S_1^{**}} + F \frac{S_1^*}{\mu b^2 S_1^{**}} \left[S_2 \left(\frac{\pi m}{d}\right)^2 + S_1^x \left(\frac{\pi n}{c}\right)^2\right] = 0$$

with the argument u = ax + bt and

$$S_1^{**} = S_1 + v^2 \frac{\mu_F \mu_F}{\mu}. \tag{III.12}$$

The exponential expression F = exp iru gives the characteristic exponents for the time

$$i r b_{12mn} = \frac{S_1^*}{S_1^{**}} \left[-\frac{\delta_0}{2\mu} \pm \sqrt{\frac{\delta_0^2}{4\mu^2} + \frac{S_1^{**}}{\mu} \left(\frac{\pi n}{c}\right)^2 + \frac{S_1^{**}}{\mu} \frac{S_2}{S_1^*} \left(\frac{\pi m}{d}\right)^2}\right], \tag{III.13}$$

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and the solution of (III.9) then has the form

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{\pi n x}{c} \sin \frac{\pi m y}{d} \left(c_{1mn} e^{-\frac{i r b_{1mn}}{w}(x - wt)} + c_{2mn} e^{-\frac{i r b_{2mn}}{w}(x - wt)}\right) \tag{III.14}$$

or

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{\pi n x}{c} \sin \frac{\pi m y}{d} \cdot \left(c_{1mn} F_{1mn}(ax + bt) + c_{2mn} F_{2mn}(ax + bt)\right) \tag{III.15}$$

with constants c_{1mn} and c_{2mn} which are again random. These constants can be established so that the initial disturbances (III.2) are satisfied. The transverse waves F_{1mn} (ax + bt) and F_{2mn} (ax + bt) alone do not satisfy the boundary conditions in x for z (x,y,t), since although the factor sin $\frac{\pi n x}{c}$ appears in the solutions of (III.14) and (III.15), these waves are distorted quasilocally or locally and the result is that the edges of the membrane remain at rest and do not undergo deflection. Solutions (III.6) and (III.14) as well as (III.15), however, display exponential dependence on the wave forms in x. Such wave forms were also given by Sparenberg [10] for an infinitely long membrane subjected to flow. Different results were obtained here, however, with respect to behavior with time. In order to determine the behavior of the solutions with time, the exponents (III.13) must be discussed by analogy with the above. It turns out that instability with vanishing damping $\delta_0 = 0$ can be avoided if

$$\begin{aligned} & S_2 < 0 \text{ and } S_1{}^* < 0 \\ \text{or } & S_2 = 0 \text{ and } S_1{}^{**} < 0 \\ \text{or } & S_2 > 0 \text{ and } S_1{}^{**} < 0 \text{ and } S_1{}^* > 0 \\ \text{or } & \sqrt{-\frac{S_1}{\mu_F}} < v < \sqrt{-\frac{S_1}{\mu_F}-\frac{\mu}{\mu_F}} . \end{aligned} \qquad \text{(III.16)}$$

On the other hand, if we assume damping with $\delta_0 > 0$, in order to avoid instability or implementation of stability we will have the requirement

$$S_1{}^* = S_1 + v^2 \mu_F < 0. \qquad \text{(III.17)}$$

We can enjoy only a slight degree of conviction in accepting criterion (III.16) in contrast to (III.17) because of its inhomogeneous structure, which includes numerous possibilities and because it also partly contains the force $S_1{}^{**}$ which cannot be explained physically. Hence, from the physical and technical standpoint, it is also unacceptable because it was developed without consideration of damping and damping forces are always present in reality. The /29 stability theory of Lyapunov offers the clue that the criterion (III.17) obtained with consideration of damping may be considered correct. Criterion (III.16) can only guarantee avoidance of instability but cannot guarantee establishment of stability, inasmuch as $\delta_0 = 0$ was assumed in establishing this criterion. However, according to Lyapunov, this is precisely the critical case, in which the theory predicts that the smallest nonlinearities which were disregarded in the present linearized calculation, can lead to stability or instability. It is therefore, in fact, uncertain which motion condition actually exists. However, if damping is taken into account in the calculation and we require avoidance of instability, stability will in fact be achieved, and the disregarded nonlinearities cannot cause any further changes in this result. The stability criterion thus obtained (III.17) is also very clearly composed addditively of the stress S_1 on the membrane and the motive power produced by the stream pressure $v^2 \mu_F$. It states that the stress on the membrane in the x-direction must necessarily consist of a pull $S_1 < 0$, for a stable motion state to be possible at all, and then allows calculation of the critical flow velocity v_k for the membrane in the form

$$v_k = \sqrt{-\frac{S_1}{\mu_F}} . \qquad \text{(III.18)}$$

The damping factor δ_0 is no longer contained in this criterion, and criterion (III.16) is not always obtained for $\delta_0 \to 0$ from (III.17) or (III.18). Hence, there is a discontinuity with respect to damping factor δ_0 in the stability criteria. Since criterion (III.17) or (III.18) exists only with consideration of damping, the present result may also be interpreted as indicating that the damping that exists when pressure law (II.2) is applied has a destabilizing effect on the oscillating membrane and ensures that flutter oscillations develop for all flow velocities $v > v_k$.

Instability phenomona, caused by critical velocities, can also develop in the case of rudders or single-bladed weather vanes. This discovery was made by Weidenhammer [11]. In such a system, with one degree of freedom, the critical velocity depends on the damping factor, while the critical velocity given by criterion (III.18) is independent of the value of the damping factor. In order to obtain unsteadiness in the stability criterion with respect to the damping factor, the system must possess at least two degrees of freedom. Ziegler [12] was the first to show the destabilizing effect of damping on a mechanical system and the resultant unsteadiness in the stability criterion for a double pendulum with accompanying load.

IV. Determination of the Weight Per Unit Area or Pressure Layer.

The calculation, as performed thus far, is based on a known weight per unit area μ_F of the flowing medium. The critical flow velocity v_k according to (V.10) is dependent on μ_F, and the value of μ_F is likewise dependent on the form of oscillation of the membrane. In the following, we shall describe a calculation of μ_F which is based on the singularity method. According to Birnbaum [13] and Glauert [14], the flow around a thin profile (Figure 3) can be obtained through interaction with a vortex layer of intensity k(x). The induced velocity $v_0(x)$ at point x, caused by the totality of interactions $k(x^1)$ at points x^1, is given by the Biot-Savart law

$$v_0(x) = \frac{1}{2\pi} \int_{x'=0}^{c} \frac{K(x')\,dx'}{(x'-x)}.$$

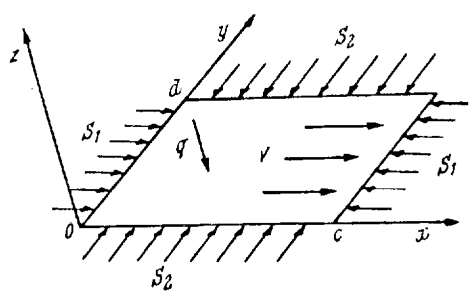

Figure 2. Membrane Plate With Rectangular Boundaries Subject to Flow.

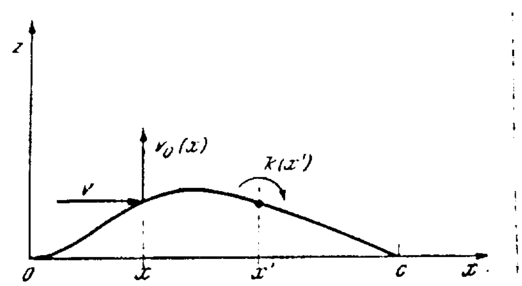

Figure 3. Flow Around a Thin Profile With Application of a Vortex Layer.

The expression

$$k(x) = 2v\left[A_0 \operatorname{ctg}\frac{\Theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\Theta\right] \qquad \text{(IV.1)}$$

for the vortex application k(x), with substitutions $x = \frac{c}{2}(1 - \cos\Theta)$ and $x^1 = \frac{c}{2}(1 - \cos\phi)$ $(0 \leq \Theta \leq \pi,\ 0 \leq \phi \leq \pi)$ with consideration of integral values

$$\int_{\varphi=0}^{\pi} \frac{\cos n\varphi\, d\varphi}{(\cos\varphi - \cos\Theta)} = \pi \frac{\sin n\Theta}{\sin\Theta} \quad (n = 0, 1, 2, \ldots)$$

and the flow condition, becomes

$$\frac{v_0(x)}{v} = \frac{dz}{dx} = -A_0 + \sum_{n=1}^{\infty} A_n \cos n\Theta. \qquad \text{(IV.2)}$$

[illegible] it is possible to determine the [illegible] this and we obtain

$$A_0 = -\frac{1}{\pi}\int_{\Theta=0}^{\pi} \frac{dz}{dx}\, d\Theta, \quad A_n = \frac{2}{\pi}\int_{\Theta=0}^{\pi} \frac{dz}{dx} \cos n\Theta\, d\Theta.$$

The lift, based on the unit length of the profile depth, is calculated according to the Kutta-Joukowski lift formula with consideration of the corresponding sign determination which applies here and the density ρ of the medium at d P/dx $q = -v\rho k(x)$. If we equate this lift to the static pressure components developed in the pressure law (II.2) by the relative centripetal acceleration (Figure 1), and use (IV.1), we will have

$$q = -2\,\varrho\, v^2 \left[A_0 \operatorname{ctg}\frac{\Theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\,\Theta\right] = v^2 \mu_F z''$$

and thus obtain for the weight per unit area

$$\mu_F = -2\,\varrho \frac{\left[A_0 \operatorname{ctg}\frac{\Theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\,\Theta\right]}{z''} \qquad \text{(IV.3)}$$

Thus, the value obtained in this fashion for μ_F can also be used for the terms in motion equation (III.9) in the sense of an approximation theory for describing the flutter oscillations. Since μ_F can also be interpreted as the weight per unit mass of a flowing medium for a tube with curvature z'' with flow through it and having a rectangular cross section of width equal to 1, division by the density ρ of the medium gives the height h of the cross section, so that $\mu_F = \rho h$. This height h can be represented as the thickness or the height of a pressure layer, in which the pressure buildup on the surface of the membrane subjected to flow is created by the flowing of the medium at the velocity v, governed by the centrifugal forces of the medium elements in the vicinity of the bent membrane. The flow around the membrane can therefore be expressed by the totality of an infinite number of tubes having cross sectional height h and infinitesimal cross sectional width dy arranged close together and parallel to the x-axis. Hence, the motion process of the oscillating membrane with flow over it is attributed primarily to the interaction between the membrane and the pressure layer or the weight per unit area μ_F by analogy with the tube with flow through it. The flow around the membrane plate corresponds to flow through a tube with a bending resistance E I = 0, or flow through a hose [15]. The weight per unit area μ_F from (IV.3) or the thickness h of the pressure layer will generally vary with the profile coordinate x or Θ. Hence, it is appropriate within the scope of an approximation method, such as is developed here to introduce an average value for μ_F which is still more exact instead of μ_F and likewise introduce an average value for the pressure layer thickness h as well. In order to obtain initially an estimate of the order of magnitude of μ_F and to ascertain the influence of constant curvature of the profile with a bulge f on the weight per unit area, we will assume for the circular profile according to Figure 4 /32

$$z = \frac{f}{2}(1 - \cos 2\Theta). \tag{IV.4}$$

which has the curvature $z'' = \frac{8f}{c^2}$, to be the result of the calculation. We will have $A_0 = 0$, $A_1 = 4\frac{f}{c}$ and $A_n = 0$ $(n = 2,3,4,\ldots)$. Hence,

$$\mu_F = \varrho c \sin \Theta, \tag{IV.5}$$

as the average weight per unit area $\bar{\mu}_F$ we will have

$$\bar{\mu}_F = \frac{1}{c}\int_{x=0}^{c} \mu_F \, dx = \frac{\pi}{4}\varrho c = 0{,}785\,\varrho c \tag{IV.6}$$

as the average value over the upwardly convex profile with profile depth c. The average thickness of the pressure layer is therefore

$$\bar{h} = \frac{\pi}{4} c = 0{,}785\, c. \tag{IV.7}$$

Determination of $\bar{\mu}_F$ and $\bar{h}$ is accomplished with particular simplicity in this case, since in (IV.3) the curvature that appears in the denominator for the profile (IV.4) is constant over the entire depth of the profile.

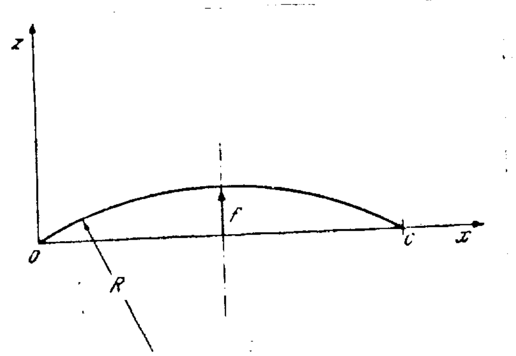

Figure 4. Circular Profile With Flow Around It as the Lowest Possible Form of Oscillation of the Membrane.

The lowest form of oscillation of the membrane plate can be approximated by the circular pattern of the deflections according to (IV.4). Higher oscillation forms can be approximated by lining up such segments of a circle with alternating signs, so that a periodic wave train results, which correspond to a sinusoidal pattern for the wave profile at higher wave numbers. This /33 approximation would have the advantage that in (IV.3) the curvature z'' would be constant for each segment and therefore taking the average over μ_F would also be simple from the calculating standpoint. In the meantime, however, on the assumption of such a wave profile, the curvature over the base c of the profile would behave in an unsteady fashion, and the calculation would prove to be quite tedious as far as determining the deflections z according to Fourier from the curve of the curvatures is concerned. Hence, for the profile

$$z = f \sin (2\lambda - 1) \frac{\pi x}{c}, \quad (\lambda = 1, 2, 3, \ldots) \tag{IV.8}$$

which describes a sinusoidal half-arc over the base c with $\lambda = 1$ and which can then be compared with the circular profile (IV.4), after which the weight per unit area μ_F can be determined. For values $\lambda > 1$ (IV.8) describes a profile with $2\lambda - 1$ half-waves. If we recall that the relationship

$$\cos (x \sin y) = I_0 (x) + 2 \sum_{n=1}^{\infty} I_{2n} (x) \cos 2 n y \tag{IV.9}$$

exists, where the $I_{2n}(x)$ are the Bessel functions of the first type and the second order, we will then obtain for the profile according to (IV.8) as a function of the coordinate Θ using (IV.9)

$$z = -f (-1)^{\lambda} \left[I_0 \left(\lambda \pi - \frac{\pi}{2} \right) + 2 \sum_{n=1}^{\infty} I_{2n} \left(\lambda \pi - \frac{\pi}{2} \right) (-1)^n \cos 2 n \Theta \right].$$

The flow condition (IV.2) then becomes

$$\frac{v_0 (x)}{v} = \frac{dz}{dx} = (-1)^{\lambda} 8 \frac{f}{c} \sum_{n=1}^{\infty} I_{2n} \left(\lambda \pi - \frac{\pi}{2} \right) (-1)^n n \frac{\sin 2 n \Theta}{\sin \Theta}.$$

If we use the relationship

$$\frac{\sin 2 n \Theta}{\sin \Theta} = 2 \sum_{i=1}^{n} \cos (2 i - 1) \Theta, \quad (n = 1, 2, 3, \ldots)$$

we will then have

$$\frac{v_0(x)}{v} = \frac{dz}{dx} =$$

$$= (-1)^\lambda\, 16\, \frac{f}{c} \sum_{m=1}^{\infty} \sum_{p=m}^{\infty} I_{2p}\left(\lambda\pi - \frac{\pi}{2}\right)(-1)^p\, p \cos(2m-1)\,\Theta =$$

$$= (-1)^\lambda\, 16\, \frac{f}{c} \sum_{m=1}^{\infty} a_{2m-1} \cos(2m-1)\,\Theta$$

with /34

$$a_{2m-1} = \sum_{p=m}^{\infty} I_{2p}\left(\lambda\pi - \frac{\pi}{2}\right)(-1)^p\, p.$$

If we compare this result with the flow condition (IV.2), we will obtain the coefficients of the vortex interaction with $A_0 = 0$, $A_n = 0$ $(n = 2,4,6,\ldots)$ and

$$z''_m = \frac{1}{(x_2 - x_1)} \int_{x=x_1}^{x_2} z''\, dx = (-1)^\lambda\, 2\pi \frac{f}{c^2} (2\lambda - 1)^2$$

For calculating the average weight per unit area μ_F we must now define an average value

$$A_n = (-1)^\lambda\, 16\, \frac{f}{c}\, a_{2m-1} = (-1)^\lambda\, 16\, \frac{f}{c} \sum_{p=m}^{\infty} I_{2p}\left(\lambda\pi - \frac{\pi}{2}\right)(-1)^p\, p. \qquad \text{(IV.10)}$$

$$(n = 1, 3, 5, \ldots)$$

for (IV.3), in which the integration extends over the central portion of the average half-wave of the wave profile (IV.8) from the point $x_1 = \frac{c(\lambda - 1)}{(2\lambda - 1)}$ to the point $x_2 - \frac{c\lambda}{(2\lambda - 1)}$. Here

$$\mu_F = -2\varrho \frac{1}{(x_2 - x_1)} \frac{\int_{x=x_1}^{x_2} \left[A_0 \operatorname{ctg}\frac{\Theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\,\Theta\right] dx}{z''_m}$$

represents the arithmetic mean over the same region of the profile (IV.8). It is attained in such a fashion that the curvature of the profile (IV.3) behaves constantly approximately like a circular wave profile. Further calculation then gives

$$\mu_F = -\frac{1}{2} \frac{\varrho c}{(x_2 - x_1)\, z''_m} \Big[A_1 (2\varepsilon + \sin 2\varepsilon) + \qquad \text{(IV.11)}$$

$$+ 4 \sum_{n=2}^{\infty} \frac{A_n}{(n^2 - 1)} \sin\frac{\pi n}{2} (n \sin n\varepsilon \cos\varepsilon - \cos n\varepsilon \sin\varepsilon)\Big]$$

for the profile (IV.8) or

$$\bar{\mu}_F = -\frac{4}{\pi}\,\frac{\varrho c}{(2\lambda - 1)}\Big[a_1\,(2\varepsilon + \sin 2\varepsilon) - \qquad \text{(IV.12)}$$

$$-\sum_{m=2}^{\infty}\frac{(-1)^m a_{2m-1}}{m\,(m-1)}\,((2m-1)\sin(2m-1)\,\varepsilon\cos\varepsilon - \cos(2m-1)\,\varepsilon\sin\varepsilon)\Big].$$

The angle ε is the complementary angle of the angles Θ_1 and Θ_2 associated /35
with boundaries x_1 and x_2,

$$\sin\varepsilon = \frac{1}{(2\lambda - 1)}$$

between which the relationships $\Theta_1 = \frac{\pi}{2} - \varepsilon$, $\Theta_2 = \frac{\pi}{2} + \varepsilon$, $x_1 = \frac{c}{2}(1 - \cos\Theta_1)$ and $x_2 = \frac{c}{2}(1 - \cos\Theta_2)$ hold. The numerical evaluation of (IV.12) for $\lambda = 1$ leads to $\varepsilon = \frac{\pi}{2}$ and then to the disappearance of the entire sum expression in (IV.11) and (IV.12), so that

$$\bar{\mu}_F = -4\,\varrho\, c\, a_1$$

remains. For this lowest half-wave number, we obtain the value of coefficients a_1 to $a_1 = -0.223$, so that

$$\bar{\mu}_F = 0{,}890\,\varrho\, c \qquad \text{(IV.13)}$$

and for the thickness of the pressure layer we will have the value

$$\bar{h} = 0{,}890\, c \qquad \text{(IV.14)}$$

The differences between the numerical values (IV.6) and (IV.7) for the circular profile and the values (IV.13) and (IV.14) for the s nusoidal profile are relatively slight according to the method used for taking the average, which represents a certain confirmation of the accuracy of this type of approximation. Figure 5 shows the manner in which μ_F is dependent on the higher half-wave numbers of the profile. In this diagram, the curve of $\mu_F/\rho c$ according to (IV.12) and therefore that of h/c over the higher half-wave numbers λ is shown.

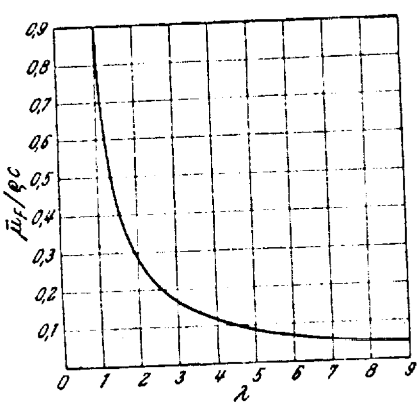

Figure 5. Dimensionless Weight Per Unit Area $\mu_F/\rho c$ or Dimensionless Pressure Layer Thickness h/c as the Function of the Wave Factor λ of the Oscillating Membrane.

The definition used for calculation of the average weight per unit volume μ_F or pressure layer thickness h found in (IV.10) contains a certain degree of arbitrariness. As indicated by a comparison of numerical values with the results of another, simplified calculation of μ_F, the defined taking of the average (IV.10) does make sense from a mechanical standpoint. If we use only the central point of profile $\theta = \frac{\pi}{2}$ on the middle half-wave for evaluating the flow process around the wave profile due to the pressure distribution, and perform the calculations for this point, for which the curvature of the profile /36

$$z'' = (-1)^{\lambda} \frac{f\pi^2}{c^2} (2\lambda - 1)^2$$

is μ_F according to (IV.3), we will have

$$\mu_F = -\frac{32}{\pi^2} \frac{\varrho c}{(2\lambda - 1)^2} [a_1 - a_3 + a_5 - a_7 + \ldots].$$

It is, however,

$$[a_1 - a_3 + a_5 - a_7 + \ldots] = -\sum_{i=0}^{\infty} I_{2+4i}\left(\lambda\pi - \frac{\pi}{2}\right)(1 + 2i)$$

and therefore the formula

$$\mu_F = \frac{32}{\pi^2} \frac{\varrho c}{(2\lambda - 1)^2} \sum_{i=0}^{\infty} I_{2+4i}\left(\lambda\pi - \frac{\pi}{2}\right)(1 + 2i), \qquad \text{(IV.15)}$$

which is much simplier in contrast to (IV.11) and (IV.12), is available for calculation of the weight per unit area or pressure layer thickness. The values calculated for μ_F in this fashion can also be used as representatives of the value $\overline{\mu}_F$. If we designate the values that follow from (IV.15) with μ^*_F and likewise those for pressure layer thickness with h* and carry out the calculation for the odd numerical values λ = 1,3,5,7 and 9 the weight per unit area and pressure layer thickness according both to (IV.12) and (IV.15), the factors of these values can be taken from the table. The maximum deviation between the numerical values occurs in the case of λ = 1. However, since the factor of μ^*_F or h* calculated according to the simplified formula (IV.15) for these lowest forms of oscilation of the membrane plate is located between the factor (IV.6) and (IV.7) for the circularly curved contour and the factor (IV.13) and (IV.14) for the sinusoidal half-wave, the values calculated in simplified fashion according to (IV.15), which are always smaller than those from (IV.12), are also acceptable. The factors calculated in those ways, in the case of even λ-values, lie slightly below the curve plotted for odd λ-values. The reason for this is the fact that in the case of even λ-values their downwash of the profile at the point $x = c$ relative to the bulge of the profile at point $x = \frac{c}{2}$ is opposed as in the case of odd values of λ. Numerically, these deviations which actually lead to a slightly s nusoidal pattern of the curve in Figure 5 and the corresponding curve which describes relationship (IV.15) (they are not indicated in Figure 5) are so small that they can remain undetected. Finally, these discrepancies are a consequence of the combined representations of the average values, which can, however, be eliminated easily by combining several half-waves of the profile in the averaging process, so that a smoothing of the curves results. For practical requirements the averagings of μ_F and h will suffice; since it is better to keep μ_F numerically larger rather than too small, there is a safety factor provided in calculating the critical velocities.

λ	1	3	5	7	9
$\bar{h}/c;\ \bar{\mu}_F/\varrho c$	0,89037	0,16187	0,08439	0,05827	0,04373
$h^*/c;\ \mu^*_F/\varrho c$	0,81256	0,15065	0,08084	0,05488	0,04144

V. Prospects.

The calculation of the critical flow velocities v_k of the membrane according to (III.18) is now possible using the weight per unit area μ_F which we have determined. The lowest critical velocity v_k is obtained for the lowest form of oscillation of the membrane with a half-wave as a deflection for which (according to IV.13) $\mu_F = 0.890\ \rho c$, so that

$$v_k = \sqrt{-1{,}124\,\frac{S_1}{\varrho c}} = 1{,}06\sqrt{-\frac{S_1}{\varrho c}} \qquad \text{(V.1)}$$

Higher critical velocities are given higher wave numbers for the membrane. Thus, for example, for $\lambda = 3$, i.e., with 5 half-waves of the profile, the factor in (V.1) under the radical is 6.173 according to (IV.12) (Table), so that

$$v_k = \sqrt{-6{,}173\,\frac{S_1}{\varrho c}} = 2{,}48\sqrt{-\frac{S_1}{\varrho c}} \qquad \text{(V.2)}$$

For $\lambda = 2$, in other words, for 3 half-waves of the profile, we obtain $\mu_F = 0.275$ ρc from Figure 5, and thus obtain a value of 1.91 for the factor in front of the radical in (V.1) and (V.2). Assignment of the wave numbers to the various appropriate critical velocities is therefore possible in theory with the aid of Figure 5, from which the factors of the $\mu_F/\rho c$ values can be taken. In the case of even half-wave numbers, the calculation was not performed. It could be done, however, with the analogous expression $z = f \times \sin 2\lambda \frac{\pi x}{c}$ $(\lambda = 1,2,3,\ldots)$ corresponding to (IV.8). Instead, the interpolation can be carried out to give the factors in front of the radical sign in the stability criterion (IV.1) or also (IV.2) with sufficient accuracy. Thus, the value of the factor for 2 half-waves of the profile $\frac{1}{2}$ (1.06 + 1.91) = 1.49 and for four half-waves $\frac{1}{2}$ (1.91 + 2.48) = 2.20. However, formula (V.1) may be significant for application, because one can calculate from it the lowest, indeed possibly critical velocity v_k and achieve stability of the membrane for all flow velocities v below v_k according (V.1).

Recently, Thwaites [16], Nielsen [17], and Heynatz and Zierep [18] investigated the stability characteristics of sails or membranes in a static fashion, i.e., without consideration of supporting members under the assumption that resting, critical wave shapes of the sails result which have critical flow velocities[2]. From [16] and [17], we can obtain the factor for the lowest critical velocity in (V.1), 1.07; the factor for 5 half-waves of the profile is 2.73. The table shows the factors calculated in the present paper on

Half-Wave Numbers	1	2	3	4	5
Factors	1,06	1,49	1,91	2,20	2,48
Factors=	1,07	1,66	2,08	2,42	2,73

[2] The kinetic stability criteria for such sails with edges that are laterally free are, however, the same as the criteria derived in this paper for membranes with laterally fastened edges. This follows from the fact that the solution (III.4) for the motion equation (III.9) then contains the cosine function instead of the sine function in y.

the first line and factors taken from [16] and [17] on the second line[3]. The fact that even at five half-widths the difference between the numerical values of 2.73 and 2.48 is still very slight constitutes a confirmation of the theory described in this paper. In mechanics, it amounts to the agreement of the static with the kinetic stability criteria for the bending rod. In the present case, the situation is somewhat more complex, since the inclusion of the supporting member and the damping in the motion equation (III.9) gives a correct result only in conjunction with the proper pressure law (II.2). The calculated results indicate that the pressure law (II.2) is physically correct. It is not without interest from the mathematical standpoint that even with a more general pressure law which is composed additively from (II.2) and (II.3) the calculation again yields criterion (III.18). As far as the derivation of this criterion is concerned, it is important to have inclusion of the inertial term in the motion equation of the membrane caused by the Coriolis acceleration $2\ v\ z_{xt}$. /39

The flutter oscillations can be seen particularly well in the case of ribbons or flags. Figure 6 shows a photograph of a ribbon in a wind tunnel. One can clearly see the instantaneous sinusoidal pattern, which changes with time, of the deflections of the membrane (cloth) during flow. This wave pattern travels during flutter along the cloth with the wave velocity w that follows from (III.11). For the wave velocity, using v_k according to (III.18), we will obtain from (III.11) the simple relationship

$$w = v - \frac{v_K^2}{v}, \qquad \text{(V.3)}$$

which expresses the wave velocity w as a function of the flow velocity v and the appropriate critical velocity v_k. In the case of all flow velocities $v < v_k$ (subcritical or stable condition) the waves travel at a wave velocity $w < 0$ against the direction of the flow velocity v through the membrane and for all $v > v_k$ (supercritical or unstable state) the transverse waves travel with a wave velocity $w > 0$ in the direction of the flow v through the membrane.

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[3] The author would like to express his particular appreciation to Prof. Dr. Zierep for this reference in the literature after the work had been completed, thus enabling this comparison between numberical values to be made subsequently.

Figure 6. Photograph of a Fluttering Cloth in a Wind Tunnel

In the first case, nothing can be seen of the flutter, since the slightest disturbances are immediately damped. In the second case, the travel of the transverse waves can be seen clearly. However, in the fluttering membrane the same types of phenomona can be seen as in an oscillating drive belt, which served in [15] as a special case of a tube with flow through it. In particular, the test in the wind tunnel also revealed that as the flow velocity v increases the wave velocity w likewise increases and according to (V.3) $vw \rightarrow v$. Thus, in addition to the good agreement of the above described numerical values, it is a phenomenological confirmation of the theory given above. In addition, the sails of sailboats moving with the wind show transverse oscillations of the type under discussion. These flutter oscillations are not caused by some disturbing objects in the airstream, such as the mast supporting the sail of a sailboat, which could create vortices and thus stimulate the flutter oscillations, but they arise solely from the supply of energy in the airstream. An energy balance based on the fluttering system composed /40 of the membrane is not possible, since any amount of energy can be drawn from a medium flowing with a constant flow velocity v. This is a typical feature of nonconservative systems, to which the membrane plate with flow over it must be assigned and which leads to the described lack of steadiness in the derivation of the stability criterion.

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